Some Remarks on the Quantization of Gauge Theories

Radhika Vathsan *

The Institute of Mathematical Sciences, C.I.T. campus, Taramani, Madras 600 113, India.

The methods of reduced phase space quantization and Dirac quantization are examined in a simple gauge theory. A condition for the possible equivalence of the two methods is discussed.

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INTRODUCTION

A gauge theory is regarded in the canonical framework as a system with first class constraints [1]. In the classical analysis according to Dirac, the Hamiltonian H is the canonical one H_c plus an arbitrary linear combination of the first class constraints ϕ_i. This means that the classical trajectories involve arbitrary functions of time: the Lagrange multipliers λ_i. So a given physical state doesn't correspond to a unique set of canonical variables on the phase space Γ. This problem can be circumvented in either of two ways:

- Gauge fixing constraints χ_i are introduced, one for each ϕ_i, such that they are preserved in time, i.e.

$$\{H, \chi_i\} \simeq 0$$

  and the matrix $C_{ij} = \{\phi_i, \chi_j\}$ is nonsingular. (This then becomes a theory with second class constraints.) Thus the λ_i's are fixed so that evolutions from initial states on the submanifold Γ^* defined by $\phi_i = \chi_i = 0$ are unique. (For future reference, we will denote by $\Gamma^{*\prime}$ the constraint surface $\phi_i = \chi_i = 0$ and provided $\det C \neq 0$ everywhere on the surface we will denote it by Γ^* and refer to it as the reduced phase space.)

- Since the λ_i's bring in the arbitrary time dependence, all points on an orbit $\mathcal{O}$ generated by the gauge generators ϕ_i must be regarded as physically equivalent. So if $\hat{\Gamma}$ is the constraint surface $\phi_i = 0$, then the true dynamical trajectories lie on $\tilde{\Gamma} \equiv \hat{\Gamma}/\sim$, where $\sim$ is the equivalence relation $P \sim P'$ if $P, P' \in \mathcal{O}$.

The surface Γ^* is diffeomorphic to $\tilde{\Gamma}$ provided the surface $\chi_i = 0$ intersects each orbit in $\hat{\Gamma}$ exactly once. This condition on the gauge-fixing constraints χ_i is a prerequisite for the equivalence of the two approaches. For the first case, the condition of invertibility of the matrix C_{ij} ensures that *locally* the $\chi_i = 0$ surface intersects $\hat{\Gamma}$ only once, but not necessarily globally. This point has bearing on the quantization of a gauge theory, since quantum theory is sensitive to the global properties of the phase space to be quantized.

These two approaches have their counterparts in the quantization of gauge theories –

Method A: Reduced Phase Space Quantization– fix the gauge to obtain the space Γ^* and define the Poisson bracket structure on this as the Dirac brackets on the original phase space Γ. Γ^* so equipped is called the reduced phase space. It can then be directly quantized, which involves the finding of a commutator algebra representation for Poisson brackets. (This process can be complicated because the reduced phase space Γ^* is not always topologically trivial.) So here one quantizes *after* reducing the phase space.

Method B: Dirac Quantization– canonically quantize the original phase space Γ (which is usually $\mathbf{R}^{2N}$) and then impose the gauge constraints as operator conditions on the physical quantum states:

$$\hat{\phi}_i \psi_{phys} = 0.$$

These are sometimes referred to as supplementary conditions. This is quantization *before* reduction.

Notice that method A depends manifestly on a choice of gauge-fixing constraints χ_i and there is a vast freedom in this choice, in general. An immediate question is whether method A applied with two different choices of the χ_i's gives equivalent quantum theories. Method B, on the other hand, is manifestly independent of any choice of gauge. If the two methods give equivalent quantum theories, then the manifest gauge-invariance of method B reflects the gauge-independence of method A applied on a class of χ_i's. The discussion of the possible equivalence of Γ^* and $\tilde{\Gamma}$ has a crucial role to play in the equivalence of the quantum theories obtained by these two methods.

These matters are illustrated in the present work in the context of a very simple toy model gauge theory.

The model considered is described in the first section of the paper. The second section deals with its quantization by method A and the third section, method B. The choice of constraints and a discussion of a condition for

the equivalence of these two methods is discussed in the fourth section. A discussion of and conclusions from the lessons learnt from the exercise comprise the fifth section. An appendix is included, giving a short review of the geometric quantization technique used in the quantization of the reduced phase space, along with the details of the calculations for the present case.

I. THE TOY MODEL

We consider the phase space $\mathbf{R}^4$ with canonical coordinates q^1, q^2, p_1, p_2 and the constraints:

$$\phi \equiv q^{1^2} + q^{2^2} + p_1^2 + p_2^2 - R^2 = 0 \tag{I.1}$$

and

$$\chi \equiv p_2 = 0 \tag{I.2}$$

Suppose we regard the constraint ϕ as the gauge generator or the first class constraint and χ as the gauge-fixing condition. The constraint surface $\Gamma^{*\prime}$ is thus the 2-sphere S^2. The matrix

$$C \equiv \{\phi, \chi\} = \begin{pmatrix} 0 & 2q^2 \\ -2q^2 & 0 \end{pmatrix} \tag{I.3}$$

is non-singular provided $q^2 \neq 0$. This immediately shows that reduced phase space cannot be $\Gamma^{*\prime}$. Let us proceed, nevertheless, and see how to obtain the true reduced phase space Γ^*.

The Poisson bracket $\{.,.\}$ on $\mathbf{R}^4$ must be modified to the Dirac bracket $\{.,.\}^*$ on the constraint surface. This is given by

$$\{f, g\}^* = \{f, g\} - \sum_{i,j} \{f, \xi_i\} C_{ij}^{-1} \{\xi_j, g\}. \tag{I.4}$$

where ξ_i is a second class constraint and $f, g \in C^\infty(\mathbf{R}^4)$. The Dirac brackets of the canonical coordinates are

$$\begin{aligned} \{q^1, q^2\} &= -\frac{p_1}{q^2} \\ \{q^1, p_1\} &= 1 \\ \{q^2, p_1\} &= -\frac{q^1}{q^2}, \end{aligned} \tag{I.5}$$

the rest being zero. Introducing the standard coordinates (θ', φ') on the sphere $S^2 \sim \Gamma^{*\prime}$, (q^i, p_i) can be parametrized as

$$\begin{aligned} q^1 &= R \sin\theta' \cos\varphi' \\ p_1 &= R \sin\theta' \sin\varphi' \\ q^2 &= R \cos\theta' \end{aligned} \tag{I.6}$$

where $0 \leq \theta' \leq \pi$ and $0 \leq \varphi' \leq 2\pi$. This is singular at $\theta' = \pi/2$ which corresponds to the singularity of the Dirac brackets (I.6) at $q^2 = 0$, at the equator of the sphere. The Dirac bracket, which is also the induced 2-form from $\mathbf{R}^4$, is

$$\{f, g\}^* = \frac{1}{R^2 \sin\theta' \cos\theta'} \left(\frac{\partial f}{\partial \theta'} \frac{\partial g}{\partial \varphi'} - \frac{\partial f}{\partial \varphi'} \frac{\partial g}{\partial \theta'} \right) \tag{I.7}$$

defines a symplectic form on $\Gamma^{*\prime}$ minus the equator: the constraint surface $\Gamma^{*\prime}$ is *not* the reduced phase space. The reason for this, as shall be demonstrated below, is that the set of points on $\Gamma^{*\prime}$ are not in 1–1 correspondence with the set of inequivalent orbits of ϕ on the surface $\hat{\Gamma} \equiv \phi = 0$. Also, reduced phase space on which the above Dirac bracket defines a symplectic structure must be obtained by a gauge-fixing condition that selects *one* point from each orbit $\mathcal{O}$. The χ of eqn.(I.2) does not satisfy this criterion. This is now shown explicitly.

The orbits $\mathcal{O}$ are the integral curves of the Hamiltonian vector fields corresponding to ϕ, which are described by the differential equations

$$\dot{x} = \{x, \phi\}$$

$$\Rightarrow \quad \dot{q}_i = 2p_i, \quad \dot{p}_i = -2q_i, \quad i = 1, 2. \tag{I.8}$$

The general solution is

$$q_i(t) = A_i \cos(2t - \alpha_i), \quad p_i(t) = -A_i \sin(2t - \alpha_i), \tag{I.9}$$

with $A_i > 0$, i.e. circles of radii A_i in the q_i–p_i planes, with initial conditions specified by the four parameters (A_i, α_i). Not all such sets specify distinct orbits: if the set (A_i, α_i) lies on the orbit generated from the set (A_i', α_i') then the two sets describe the same orbit. This happens when $A_i = A_i' \neq 0$ and $\alpha_i' - \alpha_i = 2\tau - 2n\pi$ for some $t = \tau$, for each i. If either of the A_i's is zero then there is only one orbit. So distinct orbits can be represented by

$$\begin{aligned} q^1 &= A_1 \cos(2t - \varphi), & p_1 &= -A_1 \sin(2t - \varphi), \\ q^2 &= A_2 \cos(2t), & p_2 &= -A_2 \sin(2t), \end{aligned} \tag{I.10}$$

where $0 \leq \varphi \leq 2\pi$ and $A_i \neq 0$. (Note that $A_2 = 0$ corresponds to just *one* orbit for all values of φ.)

Now if an orbit $\mathcal{O}$ lies on $\hat{\Gamma}$ we also have $A_1^2 + A_2^2 = R^2$ so that we can write

$$A_1 = R \sin\left(\frac{\theta}{2}\right), \quad A_2 = R \cos\left(\frac{\theta}{2}\right) \tag{I.11}$$

with $0 \leq \theta \leq \pi$. The orbits lying on $\hat{\Gamma}$ are thus parametrized by the two angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$, so that the space of orbits is $\tilde{\Gamma} = S^2$.

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The reduced phase space Γ^* is obtained by a gauge choice $\chi = 0$ which cuts each of the above orbits once. The surface $p_2 = 0$ intersects the orbits (I.10) at the points $t = n\pi/2$ if $A_2 \neq 0$ and at $q_2 = 0$ for all t when $A_2 = 0$. Now $q_2 = 0$ represents one orbit, as discussed earlier. Note that this is the south pole ($\theta = \pi$) of the space of orbits $\tilde{\Gamma}$. But the other orbits are intersected *twice*, i.e. at $q_2 = \pm A_2$, corresponding to the upper and lower hemispheres of the constraint surface $\Gamma^{*'}$. This means that while the *equator* ($q_2 = 0$) maps to the south pole, *both* the hemispheres ($q_2 = \pm A_2$) map to the rest of the sphere $\tilde{\Gamma}$: this means we are double-counting. So to get the correct reduced phase space, we must restrict q_2 to be positive (say), so that θ' lies in $[0, \pi/2]$ which is the upper hemisphere alone. Now we get the reduced phase space as $\Gamma^* = S^2$, on which the Dirac bracket (I.7) actually defines the Poisson bracket–

$$\{f, g\}^* = \frac{4}{R^2 \sin\theta} \left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi} - \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\theta} \right) \tag{I.12}$$

which is the standard one on a sphere of radius $R/2$, with the usual coordinate singularity at the poles. The symplectic structure induced from that on R^4 gives the same result of course.

So the reduced phase space Γ^* of the system is S^2 after choosing as the gauge-fixing condition (I.2) *together* with the requirement that each gauge orbit is counted as cut only once.

We now proceed to quantize this system by the two methods A and B outlined in the Introduction.

II. METHOD A: QUANTIZATION OF THE REDUCED PHASE SPACE

We have here a phase space, S^2, that is not a cotangent bundle and so canonical methods of quantization cannot be applied. To quantize this, we use the technique of geometric quantization [2]. A quick review of this as well as the calculations for S^2 are provided in the appendix.

The sphere is quantizable only if the radius satisfies the Weil integrality condition: $R^2 = 2N\hbar$, $N \in \mathbf{Z}$ (cf. equation A.15 in the appendix).

Working in the complex coordinates $(z, \bar{z})$ obtained by stereographic projection through the north pole, the operator corresponding to an observable f satisfying the quantizability condition (A.29) is given by

$$\hat{f} = \frac{2\hbar}{R^2} \left(1 + |z|^2\right)^2 (\partial_{\bar{z}} f \partial_z - \partial_z f \partial_{\bar{z}})$$
$$- \left(1 + |z|^2\right) \bar{z}\partial_{\bar{z}} f + f \tag{II.1}$$

(cf. equation A.18). This acts on an $(N+1)$-dimensional

Hilbert space of sections that are locally given by polynomials in z of order at most N.

There is a natural physical interpretation of this system. The phase space S^2 can be interpreted as describing the classical dynamics of the spin degrees of freedom of a particle, represented by a vector $\mathbf{J}$ in $\mathbf{R}^3$ such that $\mathbf{J}^2 = j^2$. The magnitude of $\mathbf{J}$ is preserved and the equations of motion are understood as being of first order in the time-derivatives. The components J_1, J_2 and J_3 are given in terms of the holomorphic coordinates on the sphere of radius j by

$$J_1 = j\frac{z + \bar{z}}{1 + |z|^2},$$
$$J_2 = -ij\frac{z - \bar{z}}{1 + |z|^2}, \tag{II.2}$$
$$J_3 = j\frac{|z|^2 - 1}{1 + |z|^2},$$

and they satisfy the Lie algebra, $\{J_a, J_b\} = \epsilon_{abc}J_c$, of $SU(2)$. Upon quantizing, the integrality condition (A.15) gives

$$j = \frac{N\hbar}{2}, \quad N \in \mathbf{Z}^+, \tag{II.3}$$

and the spin operators are

$$\hat{J}_1 = \frac{\hbar}{2}[(1 - z^2)\partial_z + Nz],$$
$$\hat{J}_2 = i\frac{\hbar}{2}[(1 + z^2)\partial_z - Nz], \tag{II.4}$$
$$\hat{J}_3 = \frac{\hbar}{2}[2z\partial_z - N].$$

The Hilbert space is $(N + 1)$-dimensional. One can see that $\sum \hat{J}_i^2 = \frac{\hbar^2}{4}N(N + 2) = j/\hbar(j/\hbar + 1)$. This is therefore just the standard quantum theory of an elementary particle with spin $j/\hbar$, which can take half-integral values. One can also recover the Pauli matrices as the representation of the J_i's in the basis $(1, z, \ldots, z^N)$.

III. METHOD B: DIRAC QUANTIZATION

The constrained phase space Γ is now quantized by the Dirac method. Of the two second class constraints (I.1) and (I.2), one, in this case ϕ, is chosen to be the gauge-generating first class constraint while the other (χ in this case) is a gauge-fixing condition which plays no essential part in this scheme. Now one quantizes $\mathbf{R}^4$ by the canonical method, i.e., by the association of operators

$$q^a \rightarrow \hat{q}^a = q^a, \tag{III.1}$$
$$p_a \rightarrow \hat{p}_a = \frac{\hbar}{i}\frac{\partial}{\partial q^a}, \tag{III.2}$$

which act on a Hilbert space of square-integrable wave functions $\Psi(q^1, q^2)$. Of these only those represent physical states that are gauge-invariant. So the operator corresponding to the gauge constraint must annihilate these state vectors (supplementary condition):

$$\hat{\phi}\Psi(q^1, q^2) = 0 \quad \text{(III.3)}$$

$$\Rightarrow \left[q^{12} + q^{22} - \hbar^2 \left(\frac{\partial^2}{\partial q^{12}} + \frac{\partial^2}{\partial q^{22}} \right) - R^2 \right] \Psi = 0 \quad \text{(III.4)}$$

which gives

$$\Psi(q^1, q^2) = (const) e^{-(q^{12} + q^{22})/2} H_n(q^1) H_m(q^2) \quad \text{(III.5)}$$

with

$$R^2 = 2N\hbar, \quad \text{(III.6)}$$

where $N = n + m + 1$ and n and m are non-negative integers. The radius is thus quantized as even multiples of $\hbar$ and since for each $R^2 = 2N\hbar$ there are N possible states, the Hilbert space is N-dimensional.

The functions $f(q^a, p_a)$ in $\mathbf{R}^4$ that correspond to physical observables are those that commute with the gauge generator (the so-called first class observables in Dirac's terminology), i.e.,

$$\{\phi, f\} = 0$$

$$\Rightarrow \quad q^1 \frac{\partial f}{\partial p_1} - p_1 \frac{\partial f}{\partial q^1} + q^2 \frac{\partial f}{\partial p_2} - p_2 \frac{\partial f}{\partial q^2} = 0. \quad \text{(III.7)}$$

In the variables $z^a = q^a + i\delta^{ab}p_b$ we have

$$(z^a \partial_{z^a} - \bar{z}\partial_{\bar{z}^a})f(z^a, \bar{z}^a) = 0, \quad \text{(III.8)}$$

$$\Rightarrow \quad f(z^a, \bar{z}^b) = (z^a)^{k_a}(\bar{z}^b)^{k_b} \quad \text{(III.9)}$$

with

$$\sum_a k_a = \sum_b k_b.$$

For the corresponding quantum operators to be well-defined, considerations such as that of self-adjointness may further restrict this class.

For the sake of comparison with the results of the previous section, let us look at the $SU(2)$ algebra generated by the following combinations of quadratic operators of the type $z^i \bar{z}^j$:

$$J_1 = \frac{1}{4}(z^1 \bar{z}^2 + z^2 \bar{z}^1),$$

$$J_2 = \frac{1}{4i}(z^2 \bar{z}^1 - z^1 \bar{z}^2), \quad \text{(III.10)}$$

$$J_3 = \frac{1}{4}(z^1 \bar{z}^1 + z^2 \bar{z}^2).$$

Quantization, $J_i \to \hat{J}_i$ is achieved by $z^a \to \hat{z^a}$ and

$$\sum_a \hat{J}_a{}^2 = \frac{1}{16}\left(q^{a2} + p_a{}^2 \right)^2 + \frac{1}{4}(i\hbar)^2$$

$$= \left(\frac{R^2}{4} \right)^2 - \frac{\hbar^2}{4}$$

$$= \hbar^2 \left(\frac{N^2}{4} - \frac{1}{4} \right)$$

$$= \hbar^2 \left(\frac{N}{2} - \frac{1}{2} \right) \left(\frac{N}{2} + \frac{1}{2} \right). \quad \text{(III.11)}$$

If this corresponds to $\hbar^2 j(j+1)$, we get $j = (N-1)/2$, the standard result[1].

In the present instance, we find that Dirac quantization gives results equivalent to the quantization of the reduced phase space. In particular, the quantization of the parameter R, which resulted from the Weil integrality condition in the last section, appears here as a result of the normalizability of the wave functions.

A. Choice of Constraints

When a system with second class constraints can be regarded as a gauge theory, half the constraints can be chosen as first class and generate gauge transformations and the rest are gauge-fixing conditions. When there are only two second class constraints as in the example considered here, one may choose either as the gauge generator. The choices may give different theories. Dirac quantization requires no *a priori* criterion for the choice of the first class (gauge) constraints. Suppose, in the example considered here, that instead of ϕ one chose χ as the gauge generator and let ϕ be the gauge-fixing condition. The constraint surface $\Gamma^{*\prime}$ and the singular Dirac brackets remain unaltered. The orbits $\mathcal{O}$ are in this case the lines $q^2(t) = q^2(0) + t$, and again the gauge choice $\phi = 0$ intersects these at two places: $q^2(t) = \pm q^2(0)$. So the reduced phase space Γ^* is again an S^2: the upper hemisphere of $\Gamma^{*\prime}$ with the (singular) equator mapped to the south pole; quantization by method A is the same as before. However, in Method B, the physical states are obtained by imposing the constraint condition

$$\hat{\chi}\Psi(q^1, q^2) = 0,$$

$$\Rightarrow \frac{\partial}{\partial q^2} \Psi(q^1, q^2) = 0, \quad \text{(III.A.1)}$$

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[1] In comparison, geometric quantization gave an $(N+1)$-dimensional Hilbert space for spin $N/2$. This slight discrepancy can be rectified by incorporating the metaplectic correction to geometric quantization(see, for example, [4]), whereupon the two quantization schemes match exactly.

i.e. Ψ is a function of q^1 alone. The Hilbert space is infinite-dimensional and quantizable observables are general functions of q^1, p_1 and p_2. This quantization is manifestly different from that obtained previously. On the other hand, suppose we make a different gauge choice: $\chi' \equiv q^2 = 0$. Γ^* is the surface $q^2 = p_2 = 0$, which is R^2 and the Dirac quantization discussed in the beginning of the section gives the usual quantization on this, so that methods A and B give equivalent results. So here we see that method A gives different quantizations for different gauge choices.

This source of this 'discrepancy' can be traced to the observation made in the introduction regarding the intersection of the gauge-fixing surfaces with the gauge orbits. In the first case considered here, we carefully obtained the reduced phase space as $S^3/S^1 = S^2$, and chose a gauge that selected one 2-sphere for every R. In the second case, though we were careful in considering only one intersection of the surfaces generated by the gauge choice $\phi = 0$ with the gauge orbits $\mathcal{O}$, *not all orbits are cut*. A gauge choice that intersects *all* the gauge orbits is $\chi' = 0$. So in the former case, one was artificially truncating the true phase space by an inappropriate gauge choice, and thereby obtained a different dynamical system.

Now the Dirac quantization method makes no reference to any gauge-fixing and is determined once the gauge generators (ϕ_i's) are specified and a supplementary condition is imposed to ensure that the Hilbert space so constructed is associated with the true phase space $\tilde{\Gamma}$. Quantization of the reduced phase space Γ^* can be expected to give equivalent results only when Γ^* is diffeomorphic (symplectically) to $\tilde{\Gamma}$. So the gauge-fixing constraints χ_i must satisfy not only the condition $\det\{\phi_i, \chi_j\} \neq 0$, which ensures the selection of one point from each gauge orbit *locally*, but also that the resultant reduced phase space Γ^* be diffeomorphic to the space $\tilde{\Gamma}$ of orbits. This point may seem obvious in retrospect but in practice one may miss it get a resultant quantum theory which may be consistent but not reflect gauge-independence.

IV. DISCUSSION AND CONCLUSIONS

In the context of a simple gauge theory, viz. S^2 as a phase space, we have analyzed and compared two methods of quantization, viz. quantization of the reduced phase space and Dirac quantization, and examined a condition for their equivalence.

Another observation refers to quantizability itself. As is well known, in geometric quantization there exists a condition on the phase space for quantizability: the Weil integrality condition must be satisfied if the pre-quantum bundle is to exist. In the case of S^2 this restricts the radius to discrete values. This slightly counter-intuitive result is not merely a peculiarity of the geometric approach. As shown in the present example, this quantizability condition reappears though in a different guise– it is a result of the physical Hilbert space being well defined (square-integrability of the wave functions). This shows that the quantizability condition is related to the global topological properties of the phase space.

The comparison of quantizable observables shows that there exists a restricted class of classical observables that can be consistently quantized. Conditions of self-adjointness and operator commutativity with the gauge generators must also hold rigorously.

Gauge theories are encountered in many contexts and in particular cases, either method of quantization may prove convenient. We have not considered the possible difficulties in applying either of these methods, but assuming they have been tided over, one needs to be careful about capturing the true phase space of the Dirac approach (method B) in the reduced phase space of method A. Further, restrictions may be encountered in parameters entering the theory via the constraints, for example R in the present example.

APPENDIX

This is a brief review of geometric quantization and its application to the quantization of S^2.

The classical phase space Γ is a $2n$-dimensional symplectic manifold. The symplectic form ω defines a Poisson algebra $\mathcal{A}$ of observables, which are C^∞ functions on Γ. In formulating a quantization, i.e. a map from $\mathcal{A}$ to the set $\mathcal{Q}$ of operators acting on a Hilbert space $\mathcal{H}$, the basic guidelines were spelt out by Dirac:

- The map $\mathcal{A} \to \mathcal{Q}$ is linear.

- • Constants are mapped to multiples of the identity operator.

- • • For classical observables $f_i \in \mathcal{A}$ and the corresponding quantum operators $\hat{f}_i \in \mathcal{Q}$,

$$[\hat{f}_1, \hat{f}_2] = k\hat{f}_3, \qquad f_3 = \{f_1, f_2\}$$

where $\{.,.\}$ is the Poisson bracket and $[.,.]$ is the commutator. k is some constant, canonically $i\hbar$.

Geometric quantization typically achieves this in two stages. The first stage, called 'prequantization' involves finding such a map. The prequantum Hilbert space is, however, too large to be a physically reasonable quantum description. The wave functions depend on all the phase space variables, so that the standard Schrödinger description is not obtained even in the case of $\mathbf{R}^{2N}$. Also, group

representations of elementary systems turn out to be reducible. Hence we need stage two of geometric quantization, which is the choice of a polarization of the manifold.

The 'prequantum' operator corresponding to $f \in C^\infty(\Gamma)$ is constructed as follows: in local Darboux coordinates, (q^a, p_a),

$$\omega = dp_a \wedge dq^a \tag{A.1}$$
$$= d(p_a dq^a) \tag{A.2}$$

so that the symplectic potential is

$$\theta = p_a dq^a. \tag{A.3}$$

The Hamiltonian vector field X_f corresponding to f, is

$$X_f = \frac{\partial f}{\partial p_a}\partial_{q^a} - \frac{\partial f}{\partial q^a}\partial_{p_a} \tag{A.4}$$

and

$$\theta(X_f) = p_a \frac{\partial f}{\partial p_a}. \tag{A.5}$$

Then the operator representation of f is

$$\hat{f} = -i\hbar X_f + \theta(X_f) + f. \tag{A.6}$$

This acts on sections of a complex line bundle $\mathcal{B}$ over Γ, the connection potential on which is $\theta/\hbar$ (the curvature being $\omega/\hbar$). A compatible Hermitian structure $(.,.)$ must be defined on it. Now,

$$\hat{f} = -i\hbar\nabla_{X_f} + f. \tag{A.7}$$

where $\nabla_{X_f} = X_f - \frac{i}{\hbar}\theta(X_f)$. Such a line bundle exists if and only if ω satisfies the Weil integrality condition [3]. This might restrict the class of classical phase spaces that can be quantized in this approach. One way of stating this condition, for a simply connected manifold, is that the integral of $\omega/\hbar$ over any closed, oriented 2-dimensional submanifold of Γ is an integral multiple of 2π.

Stage two of geometric quantization involves the choice of a polarization P of the manifold. Sections of $\mathcal{B}$ constant along the polarization form the quantum Hilbert space $\mathcal{H}_{\mathcal{Q}}$. If the vector fields X_m are tangent to P at a point m on Γ, then a section $s : \Gamma \to \mathcal{B}$ is said to be polarized if

$$\nabla_{X_m} s = 0. \tag{A.8}$$

Thus s is a function of only n variables. So if (s, s) is the Hermitian product on $\mathcal{B}$, the Hilbert space $\mathcal{H}_{\mathcal{Q}}$ consists of polarized sections s such that

$$\langle s, s \rangle = \int_\Gamma (s, s)\, \omega^n\, < \infty. \tag{A.9}$$

In this scheme, only those observables can be directly quantized that preserve the polarization : if s is a polarized section, so must be $\hat{f}s$, meaning if X is a vector field tangent to P then we must have

$$\nabla_X(\hat{f}s) = f(\nabla_X s) - i\hbar\nabla_{[X,X_f]}s = 0,$$

i.e. $[X, X_f]$ must also be tangent to P.

Further refinements such as half-density quantization and metaplectic corrections are not considered here as this level is sufficient for the case in hand.

This is applied to S^2 in the following.

The 2-sphere is a symplectic manifold on which the measure, in spherical coordinates (θ, ϕ),

$$\omega = \frac{R^2}{4}\sin\theta d\theta \wedge d\phi \tag{A.10}$$

(where $R/2$ is the radius) serves as the symplectic form corresponding to the Poisson bracket (I.12). It is more convenient to look upon S^2 as a Kähler manifold with holomorphic coordinates z_i obtained by stereographic projection through the north (south) poles:

$$z_n = \cot\left(\frac{\theta}{2}\right)e^{i\phi}, \qquad z_s = \tan\left(\frac{\theta}{2}\right)e^{-i\phi}.$$

So it is covered by two charts U_n and U_s both isomorphic to the complex plane $\mathbf{C}$. Note that $z_n = 1/z_s$. Working in the northern chart, $z \in U_n \approx \mathbf{C}$, the symplectic form is

$$\omega_n = -i\frac{R^2}{2}\left(1 + |z|^2\right)^{-2} dz \wedge d\bar{z}. \tag{A.11}$$

The symplectic potential is

$$\theta_n = -i\frac{R^2}{2}\left(1 + |z|^2\right)^{-1} \bar{z}dz. \tag{A.12}$$

The prequantum line bundle $\mathcal{B}$, which is locally $U_i \times \mathbf{C}$, must have a curvature $\omega/\hbar$. This means that the transition function c_{ns} (on the overlap $U_n \cap U_s$) must be given by

$$\theta_s - \theta_n = i\hbar\, d\ln c_{ns}, \tag{A.13}$$

which gives

$$c_{ns} = z^{R^2/2\hbar}. \tag{A.14}$$

This is well defined only for $R^2/2\hbar \in \mathbf{Z}$. This of course is the Weil integrality condition :

$$\int_{S^2} \omega/\hbar = 2N\pi \quad \Rightarrow \quad R^2 = 2N\hbar, \quad N \in \mathbf{Z}. \quad \text{(A.15)}$$

The Hamiltonian vector field corresponding to a function $f(z, \bar{z})$ is

$$X_f = \frac{2i}{R^2} \left(1 + |z|^2 \right)^2 (\partial_{\bar{z}} f \partial_z - \partial_z f \partial_{\bar{z}}), \quad \text{(A.16)}$$

and

$$\theta(X_f) = \left(1 + |z|^2 \right) \bar{z} \partial_{\bar{z}} f. \quad \text{(A.17)}$$

So the quantum operator corresponding to f is

$$\hat{f} = \frac{2\hbar}{R^2} \left(1 + |z|^2 \right)^2 (\partial_{\bar{z}} f \partial_z - \partial_z f \partial_{\bar{z}})$$
$$- \left(1 + |z|^2 \right) \bar{z} \partial_{\bar{z}} f + f. \quad \text{(A.18)}$$

A natural choice of polarization is the Kähler polarization [2] spanned by the Hamiltonian vector fields generated by the holomorphic coordinates

$$X_z = -\frac{2i}{R^2} \left(1 + |z|^2 \right)^2 \partial_{\bar{z}}. \quad \text{(A.19)}$$

There then exists a scalar K in the neighbourhood of each point such that the symplectic potential θ given by (A.17) can be expressed as

$$\theta = -i\partial_z K, \quad \text{(A.20)}$$

where $K = \frac{R^2}{2} \ln \left(1 + |z|^2 \right)$. This potential annihilates the vectors (A.19) i.e., $\theta(X_z) = 0$, and is said to be adapted to the polarization. The polarized sections of $\mathcal{B}$ satisfy

$$\nabla_{X_z} s(z, \bar{z}) = 0, \quad \text{(A.21)}$$
$$\Rightarrow \partial_{\bar{z}} s = 0. \quad \text{(A.22)}$$

So $\mathcal{H}_{\mathcal{Q}}$ consists of holomorphic sections of $\mathcal{B}$. In order that the wave functions be well defined on all of S^2, they must be well defined in the overlap region $U_n \cap U_s$, where they are related by

$$\psi_n = c_{ns} \psi_s$$

i.e.,

$$\psi_s \left(\frac{1}{z} \right) = z^{-N} \psi_n(z). \quad \text{(A.23)}$$

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[2] On S^2 there exist no real polarizations.

So $\psi(z)$ must be a polynomial in z of order at most N. $\mathcal{H}_{\mathcal{Q}}$ is therefore spanned by the set $(1, z, \ldots, z^N)$ and is $(N+1)$-dimensional. Given the scalar K of (A.20), the Hermitian structure on $\mathcal{B}$ can then be chosen to be

$$(s, s) = \bar{s}s e^{-K/\hbar}. \quad \text{(A.24)}$$

So the inner product on the Hilbert space is given by

$$\langle \psi, \psi \rangle = \int_{\mathbf{C}} \bar{\psi}\psi \left(1 + |z|^2 \right)^{-N} \omega$$
$$= \int_{\mathbf{C}} \bar{\psi}\psi \left(1 + |z|^2 \right)^{-N-2} dz d\bar{z}. \quad \text{(A.25)}$$

A similar result holds for the chart U_s.

Quantizable observables $f(z, \bar{z})$ in this scheme must satisfy $[X_z, X_f] \in P$. Now

$$[X_z, X_f] = \frac{2i}{R^2} \frac{\partial}{\partial z} \left(\left(1 + |z|^2 \right)^2 \frac{\partial f}{\partial \bar{z}} \right) X_z$$
$$+ \frac{2i}{R^2} \frac{\partial}{\partial \bar{z}} \left(\left(1 + |z|^2 \right)^2 \frac{\partial f}{\partial \bar{z}} \right) X_{\bar{z}}. \quad \text{(A.26)}$$

For this to belong to the polarization, the second term must vanish, i.e.,

$$\frac{\partial}{\partial \bar{z}} \left(\left(1 + |z|^2 \right)^2 \frac{\partial f(z, \bar{z})}{\partial \bar{z}} \right) = 0. \quad \text{(A.27)}$$

$$\Rightarrow f(z, \bar{z}) = \frac{h_1(z) + \bar{z} h_2(z)}{1 + |z|^2}. \quad \text{(A.28)}$$

where h_1 and h_2 are functions of z alone. For such an observable to be well-defined on all of S^2, one requires that it have the same value in both charts:

$$\frac{h_1(z) + \bar{z} h_2(z)}{1 + |z|^2} = \frac{h_1'(z') + \bar{z}' h_2'(z')}{1 + |z'|^2}$$

which gives

$$h_1'(z) = z h_2(1/z)$$

$$h_2'(z) = z h_1(1/z)$$

where the prime denotes the southern chart. These must be well defined for all z. Further, restriction to real functions alone gives a general observable the form:

$$f(z, \bar{z}) = \frac{a + bz + \bar{b}\bar{z} + c|z|^2}{1 + |z|^2} \quad \text{(A.29)}$$

where the constants a and c are real and b is complex. Dynamics in this theory can be dictated by a Hamiltonian chosen from this class of observables. This completes the quantization of S^2.

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[1] P.A.M.Dirac, *Lectures on Quantum Mechanics*, Yeshiva University (1964);
M.Henneaux and C.Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (1992).

[2] N.M.J.Woodhouse, *Geometric Quantization.* 2nd edition, Clarendon, Oxford (1992).

[3] see ref [2] above or Simms and Woodhouse, *Lectures on Geometric Quantization*, Lecture Notes in Physics 53 (1976).

[4] G.M.Tuynman,"Generalized Bergman Kernels and Geometric Quantization", J.Math.Phys. **28**(3) 573 (1987).